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PG Semester I
Unit :- 4
Topic :- group theory examples

Example ① Show that the set Z of all integers form a group with respect to binary operation $*$ defined by

$$a * b = a + b + 1 \quad \forall\ a, b \in Z$$ is an abelian group.

Sol^n:— (i) Closure property:— Let $a, b \in Z$

$\Rightarrow$ $a + b + 1 \in Z$

$\Rightarrow$ $a * b \in Z$

$\Rightarrow$ Z is closed with respect to $*$

(ii) **Associativity**:— If $a, b, c \in Z$, then

$$(a * b) * c = (a+b+1) * c$$
$$= (a+b+1) + c + 1$$
$$= a + b + c + 2$$

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Also, $a * (b * c) = a * (b + c + 1)$

$= a + \{b + (c+1) + 1\}$

$= a + b + c + 2$

$\Rightarrow (a * b) * c = a * (b * c) \quad \forall\, a, b, c \in Z$

(iii) Existence of identity

An element $e \in Z$ will be the identity, if $e * a = a \quad \forall\, a \in Z$

Now, $e * a = e + a + 1$

$e + a + 1 = a \Rightarrow e = -1$

Since, $-1 \in Z$ and we have for any $a \in Z$

$(-1) * a = -1 + a + 1 = a$

$\Rightarrow -1$ is the identity element.

(iv) Existence of inverse :- If $a \in Z$, then $b \in Z$ will be the inverse of a, if $b * a = -1$ ($\because -1$ is the identity element)

Now, $b * a = -1 \Rightarrow b + a + 1 = -1$

$\Rightarrow b = -2 - a$

Also, $a \in Z \Rightarrow -2 - a \in Z$

and $(-2-a) * a = (-2-a) + a + 1 = -1$

identity ~~element~~

$\therefore (-2-a)$ is the inverse of a.

(v) Commutativity

Since, $a * b = a+b+1 = b+a+1 = b*a$

$\Rightarrow$ commutativity satisfied.

Hence, Z is an infinite abelian group under the given composition.

Example 2 :- Show that the set $\{1, -1, i, -i\}$ is an abelian finite group of order 4 under multiplication.

Soln :- (i) Closure property :- Closure property is satisfied as

$1(-1) = -1, \; 1\cdot i = i, \; i(-i) = 1,$
$i(-1) = -i$, etc.

(ii) Associativity :- Associative property is satisfied as

$(1\cdot i)(-i) = 1\cdot\{i(-i)\} = 1$
$\{1\cdot i\}\cdot(-1) = 1, \; \{i(-1)\} = -i$, etc.

(iii) Existence of identity :- Axioms on identity is satisfied, 1 being the multiplicative identity.

(iv) Existence of inverse:- Axiom on inverse is satisfied since the inverse of each element of the set exists.

$1\cdot 1 = e = 1$, $(-1)(-1) = e = 1$, $i(-i) = e = 1$

$(-i)(i) = e = 1$

(v) Commutativity:- The commutative law is also satisfied as

$1(-1) = (-1)\cdot(1)$

$(-1)i = i(-1)$, etc.

Since there are four elements in the given set, hence it is a group of order 4.